Resit exam Linear Algebra II
Friday 28/06/2024, 15:00–17:00

1 (9 = 2 + 2 + 4 + 1 pts) **Subspaces, bases and linear transformations**

Let $\mathcal{V}$ be the $\mathbb{R}$-vector space of all continuously differentiable functions $f : \mathbb{R} \to \mathbb{R}$. Here, addition of functions $f, g \in \mathcal{V}$ is defined as $(f+g)(x) = f(x)+g(x)$ for all $x \in \mathbb{R}$, while scalar multiplication is defined as $(af)(x) = af(x)$ for $a \in \mathbb{R}$ and $f \in \mathcal{V}$. Let $c_1, c_2, \ldots, c_n \in \mathbb{R}$ be distinct nonzero numbers. In this exercise we will consider the set

$$\mathcal{S} := \{a_1e^{c_1x} + a_2e^{c_2x} + \cdots + a_ne^{c_nx} \mid a_1, a_2, \ldots, a_n \in \mathbb{R}\}.$$

(a) Show that $\mathcal{S}$ is a subspace of $\mathcal{V}$.

(b) Define $T_\alpha : \mathcal{S} \to \mathcal{S}$ by

$$T_\alpha(f) = f' - \alpha f,$$

where $\alpha \in \mathbb{R}$. Show that T_α is a linear transformation.

(c) Now, suppose that $b_1, b_2, \ldots, b_n \in \mathbb{R}$ are such that

$$b_1e^{c_1x} + b_2e^{c_2x} + \cdots + b_ne^{c_nx} = 0. \quad (1)$$

Prove that $b_1 = b_2 = \cdots = b_n = 0$. *Hint:* Use mathematical induction on n. For the induction step, apply T_{c_n} to both sides of (1).

(d) Is $\{e^{c_1x}, e^{c_2x}, \ldots, e^{c_nx}\}$ a basis of $\mathcal{S}$? Motivate your answer.

2 (9 = 1 + 2 + 3 + 3 pts) **dimension theorem, matrix representation**

Denote by $\mathcal{P}$ the $\mathbb{R}$-vector space of all polynomials in the variable x, with real coefficients. Define the $\mathbb{R}$-linear transformation $T\colon \mathcal{P} \to \mathcal{P}$ by $T(f) = x \cdot f - \frac{df}{dx}$. For any $n \geq 0$, let $\mathcal{P}_n$ be the $\mathbb{R}$-subspace of $\mathcal{P}$ consisting of the polynomials f of degree $\leq n$, and let $\mathcal{Q}_n$ be the $\mathbb{R}$-subspace consisting of all $f \in \mathcal{P}_n$ such that $f(0) = 0$.

(a) Show that T is injective.

(b) The n-th Hermite polynomial $H_n(x)$ is defined as $H_n(x) = T^n(1)$.

  (i) Compute $H_1(x)$ and $H_4(x)$.

  (ii) Show that $H_{n+1}(x) = xH_n(x) - H_n'(x)$.

(c) Determine (with proof!) dim $T(\mathcal{Q}_n)$.

(d) For suitable bases of $\mathcal{Q}_2$ and of $\mathcal{P}_3$, construct the matrix of T considered as a map from $\mathcal{Q}_2$ to $\mathcal{P}_3$.

3 $(9 = 3 + 3 + 3$ pts$)$ **inner product space, orthogonal complement, adjoint**

In the $\mathbb{R}$-inner product space V with inner product $\langle \cdot , \cdot \rangle$ we fix a vector v such that $||v|| = 1$. The $\mathbb{R}$-linear transformation $R\colon V \to V$ is defined by $R(w) = w - 2\langle w, v\rangle v$.

(a) Prove that $v^{\perp} = \mathrm{Ker}(R - id_V)$.

(b) Show that the adjoint of R is R itself.

(c) Show that $||R(w)|| = ||w||$ for every $w \in V$.

4 $(9 = 3 + 4 + 2$ pts$)$ **Singular value decomposition**

Consider the matrix

$$A = \begin{bmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{6} \\ \sqrt{2} & \sqrt{2} \end{bmatrix}.$$

(a) Compute the singular values of A.

(b) Compute a singular value decomposition of A.

(c) Compute the best rank 1 approximation of A.